

M-theory compactification, fluxes and AdS_4

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Abstract

We analyze supersymmetric solutions of M-theory based on a seven-dimensional internal space with $\text{SU}(3)$ structure and a four-dimensional maximally symmetric space. The most general supersymmetry conditions are derived and we show that a non-vanishing cosmological constant requires the norms of the two internal spinors to differ. We find explicit local solutions with singlet flux and the space being a warped product of a circle, a nearly-Kähler manifold and AdS_4 . The embedding of solutions into heterotic M-theory is also discussed.

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1 Introduction

In trying to understand the structure of M-theory and string theory there has been much effort put into finding supersymmetric solutions to the classical bosonic equations of motion. As has been known for some time the requirement of supersymmetry puts a strong restriction on the form that solutions can take [1, 2, 3]. More recently it has been suggested that G-structures, obtained from spinor bi-linears, may be used as a way of classifying supersymmetric solutions [4]. Since then there has appeared work using these ideas to find all supersymmetric solutions to various supergravities theories [5]–[8], as well as work classifying solutions [9]–[23] and clarifying the nature of the supercharges [24].

In this paper we aim to revisit the subject of supersymmetric solutions of M-theory based on an internal space with $SU(3)$ structure and a four-dimensional maximally symmetric space of Lorentzian signature [17], that is, Minkowski space or AdS_4 . The Ansatz we use allows for general fluxes and we use the machinery of G-structures to analyze the system, constructing p -forms out of spinor bi-linears. However, unlike in Ref. [17], we allow the norms of the two internal spinors which define the $SU(3)$ structure to be different. The general supersymmetry conditions are then derived for this setup, showing that a non-zero cosmological constant requires the norms of the two spinor to be different. More specifically, it turns out that the cosmological constant acts as a source for the particular spinor bi-linear which governs the relative normalization of the spinors. We also find that the external flux as well as one of the three singlets of the internal flux have to be zero in general. Moreover, for a non-zero cosmological constant we conclude that the vector part of the internal flux must vanish and that the internal space can be sliced into six-dimensional half-flat manifolds.

We apply our general results to the case of internal singlet flux and derive solutions which, in general, are warped products of a circle, a six-dimensional nearly-Kähler manifold and AdS_4 . We also find a special case where the nearly-Kähler manifold degenerates to a Calabi-Yau three-fold, and a nearly-Kähler example with vanishing cosmological constant but non-trivial spinor norms.

The plan of the paper is as follows. In Section 2, we set up the Killing spinor equation of 11d supergravity and outline our conventions, Section 3 introduces the general Ansatz for the solutions considered in this paper, derives the supersymmetry conditions and analyzes their general properties and in Section 4 we study solutions with $SU(3)$ singlet flux. Finally, in Section 5, we discuss the possible embedding of solutions into Horava-Witten theory and, as a consistency check, we verify that the approximate solutions of Ref. [18, 19] satisfy our equations. Two Appendices summarize technical information: Appendix A contains gamma-matrix and spinor conventions in $D = 4, 7, 11$; and Appendix B the formalism for $SU(3)$ structures in seven dimensions.

2 Basic equations and conventions

We will work with the following bosonic equations of motion of 11-dimensional supergravity

$$R_{MN} = \frac{1}{12} \left(F_{MPQR} F_N{}^{PQR} - \frac{1}{12} g_{MN} F^2 \right), \quad (2.1)$$

$$d \star F = -\frac{1}{2} F \wedge F, \quad (2.2)$$

where F is the four-form field strength. Indices $M, N, \dots = 0, \dots, 9, \natural$ are curved 11-dimensional indices and we denote their flat counterparts by an underlined version, that is $\underline{M}, \underline{N}, \dots = 0, \dots, 9, \natural$. The above equations of motion should be supplemented by the Bianchi identity

$$dF = 0. \quad (2.3)$$

A solution of the above system of equations (2.1)–(2.3) is supersymmetric if an 11-dimensional Majorana spinor ¹ η exists which satisfies the Killing spinor equation

$$\nabla_M \eta + \frac{1}{288} \left[\Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR} \right] F_{NPQR} \eta = 0, \quad (2.4)$$

where we use the standard covariant derivative,

$$\nabla_M \eta = \partial_M \eta + \frac{1}{4} \omega_M^{\underline{MN}} \Gamma_{\underline{MN}} \eta, \quad (2.5)$$

for spinors. Conversely, a solution of the Killing spinor equation (2.4) which, in addition, satisfies the F equation of motion (2.2) and the Bianchi identity (2.3) also satisfies the Einstein equation and is, hence, a supersymmetric solution of the theory. This follows from the integrability of the Killing spinor equation [11].

3 Dimensional reduction

In this paper we are interested in supersymmetric solutions of 11-dimensional supergravity which are maximally symmetric in $3+1$ dimensions, and have an “internal” seven-dimensional space with $SU(3)$ structure. This implies that the metric is a warped product of a four-dimensional maximally symmetric space-time (Minkowski space or AdS_4) with metric $ds_{(4)}^2$ and a Euclidean seven-dimensional space with $SU(3)$ structure and metric $ds_{(7)}^2$. We denote four-dimensional coordinates by $\mathbf{x} = (x^\mu)$ with curved indices $\mu, \nu, \dots = 0, 1, 2, 3$ and their flat counterparts $\underline{\mu}, \underline{\nu}, \dots = 0, 1, 2, 3$. Seven-dimensional coordinates $\mathbf{y} = (y^m)$ are labeled by curved indices $m, n, \dots = 4, \dots, 9, \natural$ and their flat cousins are underlined, as usual. The 11-dimensional metric can then be written as

$$ds_{(11)}^2 = e^{2A(\mathbf{y})} ds_{(4)}^2 + e^{2B(\mathbf{y})} ds_{(7)}^2, \quad (3.6)$$

where $A = A(\mathbf{y})$ is the warp factor and $B = B(\mathbf{y})$ is a gauge degree of freedom which will be fixed later by a convenient choice. The most general Ansatz for the four-form F compatible with our basic assumptions is given by

$$F_{\mu\nu\sigma\rho} = f(\mathbf{y}) \epsilon_{\mu\nu\sigma\rho}, \quad (3.7)$$

$$\hat{F}_{mnpq} = \text{arbitrary}, \quad (3.8)$$

where $f = f(\mathbf{y})$ is an arbitrary function of the internal coordinates and \hat{F} denotes the internal part of the form F .

In order to manipulate the Killing spinor equation we need to decompose the 11-dimensional gamma-matrices. Using the four- and seven-dimensional gamma-matrices $\{\gamma_{\underline{\mu}}\}$ and $\{\gamma_{\underline{m}}\}$ introduced in Appendix A we define their “curved” counterparts $\{\gamma_\mu\}$ and $\{\gamma_m\}$ with respect to the vielbeins associated to $ds_{(4)}^2$ and $ds_{(7)}^2$, respectively. The 11-dimensional gamma-matrices can then be written as

$$\Gamma_\mu = e^A (\gamma_\mu \otimes 1), \quad \Gamma_m = e^B (\gamma \otimes \gamma_m), \quad (3.9)$$

where γ has been defined in Eq. (A.3).

Finally, we need to find the most general Ansatz for the spinor η in our case. The main building blocks for this Ansatz are the two seven-dimensional Majorana spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$ which exist on

¹For a summary of our gamma-matrix and spinor conventions see Appendix A.

the internal space owing to the SU(3) structure². The most general Ansatz for η based on these two spinors is given by

$$\eta = \theta_{(1)} \otimes \epsilon_{(1)} + \theta_{(2)} \otimes \epsilon_{(2)}, \quad (3.10)$$

where $\theta_{(1)}$ and $\theta_{(2)}$ are two four-dimensional Majorana spinors. These two four-dimensional spinors can be related by

$$\theta_{(2)} = (C + D_\mu \gamma^\mu + \frac{1}{2} E_{\mu\nu} \gamma^{\mu\nu} + i F_\mu \gamma \gamma^\mu + i G \gamma) \theta_{(1)}, \quad (3.11)$$

where C , D_μ , $E_{\mu\nu}$, F_μ and G are functions of \mathbf{y} . Due to maximal symmetry D_μ , $E_{\mu\nu}$ and F_μ must vanish and we are left with

$$\theta_{(2)} = (C + i G \gamma_5) \theta_{(1)}. \quad (3.12)$$

After a suitable redefinition of $\epsilon_{(1)}$ and $\epsilon_{(2)}$ the spinor Ansatz can then be written as

$$\eta = \theta \otimes \epsilon_{(1)} + i \gamma \theta \otimes \epsilon_{(2)}, \quad (3.13)$$

with a four-dimensional Majorana spinor θ . This has the same form as the spinor Ansatz in Refs. [14, 17, 20] except for one crucial difference - we do not set the norms of $\epsilon_{(1)}$ and $\epsilon_{(2)}$ equal, that is, in the language of Appendix B, we allow λ to be different from one. It is, of course, possible to work with spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$ of the same norm but then we would have to introduce an arbitrary function in the Ansatz (3.13) in order not to lose any generality. In the following, we will work with the spinor Ansatz (3.13) and unconstrained spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$. As we will see this has important consequences when we analyze the conditions for unbroken supersymmetry.

To complete the spinor Ansatz we need to specify the properties of the four-dimensional Majorana spinor θ . In the case of Minkowski space θ is, of course, simply a constant spinor. In the case of AdS₄, we start with the Killing spinor equation

$$\nabla_\mu \theta = -\frac{i}{2} \Lambda_1 \gamma_\mu \gamma \theta + \frac{1}{2} \Lambda_2 \gamma_\mu \theta, \quad (3.14)$$

where Λ_1 and Λ_2 are cosmological constants and ∇_μ denotes the covariant derivative with respect to $ds_{(4)}^2$. They imply that the Ricci tensor associated to $ds_{(4)}^2$ is given by

$$R_{\mu\nu} = -3(\Lambda_1^2 + \Lambda_2^2) g_{\mu\nu}. \quad (3.15)$$

It is known [29, 17] that spinors θ which satisfy Eq. (3.14) exist for the two cases $\Lambda_1 \neq 0$, $\Lambda_2 = 0$ and $\Lambda_1 = 0$, $\Lambda_2 \neq 0$. We will cover both cases (as well as possible more general ones) by keeping Λ_1 and Λ_2 arbitrary and working with the full Eq. (3.14). Also note that the cosmological constants Λ_i can be functions of the internal coordinates \mathbf{y} , a possibility we shall allow for as well.

We are now in a position to evaluate the Killing spinor equation (2.4) for our Ansatz. For the “internal” and “external” part we find, respectively,

$$\begin{aligned} \nabla_m \zeta_\pm &= -\frac{1}{2} \gamma_m^n \partial_n B \zeta_\pm + \frac{i}{12} f e^{B-4A} \gamma_m \zeta_\pm \\ &\mp \frac{1}{288} e^{-3B} F_{npqr} \gamma_m^{npqr} \zeta_\pm \pm \frac{1}{36} e^{-3B} F_{mnpq} \gamma^{npq} \zeta_\pm, \end{aligned} \quad (3.16)$$

$$\frac{1}{288} e^{A-4B} F_{mnpq} \gamma^{mnpq} \zeta_\pm = \mp \frac{i}{6} f e^{-3A} \zeta_\pm \mp \frac{1}{2} e^{A-B} \partial^m A \gamma_m \zeta_\pm \pm \frac{i}{2} \Lambda_1 \zeta_\mp - \frac{1}{2} \Lambda_2 \zeta_\mp, \quad (3.17)$$

²For a review of the SU(3) structure formalism for seven-dimensional manifolds see Appendix B.

where the complex spinors ζ_{\pm} are defined by

$$\zeta_{\pm} = \epsilon_{(1)} \pm i\epsilon_{(2)} . \quad (3.18)$$

Eqs. (3.16) and (3.17) must be satisfied if a solution is to be supersymmetric and we will analyze these equations in detail below.

3.1 Internal relations.

We begin by re-writing the internal equation (3.16) in terms of the forms (B.15)–(B.20) constructed as spinor bi-linears. A straightforward but somewhat tedious calculation leads to

$$d(e^{-A}\Xi) = 0, \quad (3.19)$$

$$e^{-2A}d(e^{2A}\tilde{\Xi}) = -2i\Lambda_1 e^{-A+B}\Xi_{(1)} - 2\Lambda_2 e^{-A+B}\Xi_{(1)}, \quad (3.20)$$

$$d(e^{A+B}\Xi_{(1)}) = 0, \quad (3.21)$$

$$e^{-3A-2B}d(e^{3A+2B}\Xi_{(2)}) = -e^{A-3B} * F + 3\Lambda_1 e^{-A+B}\tilde{\Xi}_{(3)+} - 3\Lambda_2 e^{-A+B}\tilde{\Xi}_{(3)-}, \quad (3.22)$$

$$e^{-2A-3B}d(e^{2A+3B}\tilde{\Xi}_{(3)-}) = 2\Lambda_1 * \Xi_{(3)}, \quad (3.23)$$

$$e^{-2A-3B}d(e^{2A+3B}\tilde{\Xi}_{(3)+}) = -e^{-3B}\tilde{\Xi}F + 2\Lambda_2 e^{-A+B} * \Xi_{(3)}. \quad (3.24)$$

where the subscripts on Ξ refer to the rank of the form. The calculation is made less laborious with software designed to perform gamma matrix algebra [30]. We can draw some immediate conclusions from these relations. Given that $\Xi_{(1)}$ and $\tilde{\Xi}$ are both real, Eq. (3.20) implies that

$$\Lambda_1 = 0 . \quad (3.25)$$

Hence, only four-dimensional Killing spinors of the second type as parameterized by Λ_2 are (potentially) consistent with supersymmetry. In the following we will drop all Λ_1 terms in the above relations, and set $\Lambda_1 = 0$ in the four-dimensional Killing spinor equation (3.14).

From Eq. (3.19) we conclude that

$$\Xi = e^A , \quad (3.26)$$

up to an irrelevant constant of proportionality which we have set to unity for simplicity. It will be useful to parameterize the relative normalization λ of $\epsilon_{(1)}$ and $\epsilon_{(2)}$, as defined in Eq. (B.27), in terms of an angle χ by setting

$$\lambda = \frac{\cos(\chi)}{1 + \sin(\chi)} . \quad (3.27)$$

Note that, from Eq. (B.28), we have $|\lambda| = |\epsilon_{(1)}|/|\epsilon_{(2)}|$. Equal norms of the spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$, therefore, correspond to the values $\lambda = \pm 1$, or $\chi = n\pi$, for any interger n . From Eq. (3.26) and the definitions (B.31), (B.32), (B.14) of Ξ , $\tilde{\Xi}$ and ζ_{\pm} one finds the norms $|\epsilon_{(1)}|$ and $|\epsilon_{(2)}|$ then satisfy

$$\Xi = |\epsilon_{(1)}|^2 + |\epsilon_{(2)}|^2 = e^A, \quad (3.28)$$

$$\tilde{\Xi} = |\epsilon_{(1)}|^2 - |\epsilon_{(2)}|^2 = e^A \sin(\chi) . \quad (3.29)$$

These relations will allow us to eliminate the implicit dependence of the supersymmetry equations on the spinor norms in favour of the warp factor A and λ (or χ). We should stress here that the parameters Λ_i act as a source for the scalar $\tilde{\Xi}$ in Eq. (3.20). By setting the norms of the spinors $\epsilon_{(i)}$ to coincide, as in Ref. [17], the scalar $\tilde{\Xi}$ and, hence, the cosmological constant is forced to vanish. However, as we have mentioned earlier, it is unnecessary to set these spinor norms equal and the cosmological constant can only be kept non-zero if one allows $|\epsilon_{(1)}| \neq |\epsilon_{(2)}|$.

3.2 External relations.

We now turn to the external supersymmetry equations (3.17). Using Eq. (B.54), the spinor $\epsilon_{(2)}$ can be eliminated and, with the help of relations (B.55)–(B.58), the number of gamma matrices in each term can then be reduced to zero or one. Since $(\epsilon_{(1)}, \gamma_m \epsilon_{(1)})$ are linearly independent in spinor space one can then obtain a set of bosonic equations

$$e^{A-4B} \hat{F} \lrcorner (J \wedge J + 2\Psi_+ \wedge V) = 12 (\Lambda_2 - \lambda e^{A-B} V \lrcorner dA), \quad (3.30)$$

$$\lambda e^{A-4B} \hat{F} \lrcorner (\Psi_- \wedge V) = -2f e^{-3A}, \quad (3.31)$$

$$e^{A-4B} (J \wedge V - \Psi_-) \lrcorner \hat{F} = -2f \lambda e^{-3A} V + 6\lambda e^{A-B} dA \lrcorner J, \quad (3.32)$$

$$\lambda e^{A-4B} \left(2\Psi_+ \lrcorner \hat{F} - \hat{F} \lrcorner (J \wedge J) V - (V \lrcorner F) \lrcorner (J \wedge J) \right) = 12 (e^{A-B} dA + \lambda \Lambda_2 V). \quad (3.33)$$

which is equivalent to the external supersymmetry conditions (3.17). Here, V is a one-form, J a two-form and Ψ is a complex three-form. These forms are related to the spinor bi-linears Ξ by (B.31)–(B.36) and, together, they characterize the $SU(3)$ structure of the seven-dimensional internal space - as reviewed in Appendix B.

At this point it is convenient to decompose the four-form \hat{F} into $SU(3)$ representations, following Ref. [20]. It turns out that \hat{F} contains three $SU(3)$ singlets, denoted by Q , c_1 and c_2 ; two vectors Y and W in $(\mathbf{3}+\bar{\mathbf{3}})_{SU(3)}$; a two-form \mathcal{A} in $\mathbf{8}_{SU(3)}$; and a three-form U in $(\mathbf{6}+\bar{\mathbf{6}})_{SU(3)}$. This decomposition is explained in Appendix B and the various forms and their properties are summarized in Table 1. The explicit $SU(3)$ decomposition of \hat{F} is given in Eq. (B.63). With this decomposition of the four-form we find the external relations become

$$e^{A-4B} (4c_1 + Q) = 6 (\lambda e^{A-B} dA \lrcorner V - \Lambda_2), \quad (3.34)$$

$$2\lambda c_2 e^{A-4B} = f e^{-3A}, \quad (3.35)$$

$$e^{A-4B} (W - Y + 2c_2 V) = -f \lambda e^{-3A} V + 3\lambda e^{A-B} dA \lrcorner J, \quad (3.36)$$

$$\lambda e^{A-4B} ((4c_1 - Q)V + 2W \lrcorner J) = -6 (e^{A-B} dA + \lambda \Lambda_2 V). \quad (3.37)$$

We now make use of the internal co-ordinate freedom and set $B = 0$, for convenience. The inner product of (3.36) and (3.37) with V (using some of the relations (B.40)–(B.48)) can be compared with (3.35) and (3.34) which immediately leads to

$$f = 0, \quad c_2 = 0. \quad (3.38)$$

Hence, we conclude that the external part of the flux (3.8) as well as the singlet part c_2 of the internal flux vanish. Further, by taking the inner product of (3.36) with J and comparing with (3.37) we find the two vector parts Y and W of the internal flux are proportional. More precisely, we have

$$Y = (1 - \lambda^2)W. \quad (3.39)$$

3.3 Summary of supersymmetry conditions

It is useful to pause for a moment and summarize the supersymmetry conditions subject to the simplifications which we have found so far. We have seen from the internal equations that only the cosmological constant Λ_2 may be allowed, and that the normalization of the spinors $\epsilon_{(i)}$ can be expressed, via Eqs. (3.28), (3.29), in terms of the warp factor A and their relative normalization λ . The latter can be conveniently parameterized by an angle χ as in Eq. (3.27). From the external

equations we have found that $f = c_2 = 0$ and that Y and W are proportional, as specified by Eq. (3.39). Further, we adopt the gauge choice $B = 0$ from here on.

The external supersymmetry conditions then simplify to

$$c_1 = -\frac{1}{4}\sin(\chi)Q - \frac{3}{2}e^{-A}\Lambda_2 \quad (3.40)$$

$$dA = \frac{1}{6}\cos(\chi)QV - \frac{\lambda}{3}\sigma, \quad (3.41)$$

where σ is defined by

$$\sigma = W \lrcorner J. \quad (3.42)$$

The internal relations can be expressed in terms of the SU(3) structure (V, J, Ψ) as well, using the relations (B.31)–(B.36) as well as the spinor normalizations fixed by Eqs. (3.28), (3.29). We find

$$e^{-2A}d[e^{3A}\sin(\chi)] = -2\Lambda_2\cos(\chi)V, \quad (3.43)$$

$$d[e^{2A}\cos(\chi)V] = 0, \quad (3.44)$$

$$e^{-3A}d[e^{4A}\cos(\chi)J] = -e^A \star \hat{F} - 3\Lambda_2[\Psi_- - \sin(\chi)J \wedge V], \quad (3.45)$$

$$d[e^{3A}(\Psi_- - \sin(\chi)J \wedge V)] = 0, \quad (3.46)$$

$$e^{-2A}d[e^{3A}\cos(\chi)\Psi_+] = -e^A\sin(\chi)\hat{F} - \Lambda_2[2\sin(\chi)\Psi_+ \wedge V + J \wedge J], \quad (3.47)$$

where here, and in the following, the Hodge star refers to the seven-dimensional internal space with metric $ds_{(7)}^2$. The internal flux (3.7) now takes the form

$$\hat{F} = -\frac{1}{6}QJ \wedge J + J \wedge \mathcal{A} + [V \wedge J + (1 - \lambda^2)\Psi_-] \wedge W - c_1\Psi_+ \wedge V + V \wedge U. \quad (3.48)$$

This flux is still subject to the equation of motion (2.2) and the Bianchi identity (2.3) which can be written as

$$d[e^{4A} \star \hat{F}] = 0, \quad (3.49)$$

$$d\hat{F} = 0. \quad (3.50)$$

Note that the non-linear $F \wedge F$ term in the equation of motion (2.2) vanishes in the case at hand since there is no external flux as a result of $f = 0$. Multiplying Eq. (3.45) with e^{3A} , taking the exterior derivative and using the equation of motion (3.49) one obtains a relation which differs from Eq. (3.46) only by terms which involve $d\Lambda_2$. In fact, it is easy to see this is consistent only if

$$\Lambda_2 = \text{const}. \quad (3.51)$$

We have, therefore, concluded that the cosmological constant Λ_2 of the four-dimensional space is independent of the internal coordinates.

For later reference, it will be useful to re-write the internal supersymmetry equations (3.43)–(3.47) by solving for the exterior derivatives of the various forms involved and by eliminating c_1 and dA using the external relations (3.40) and (3.41). One finds

$$d\chi = -\left(2e^{-A}\Lambda_2 + \frac{1}{2}\sin(\chi)Q\right)V + \lambda\tan(\chi)\sigma, \quad (3.52)$$

$$dV = \lambda\left(\frac{2}{3} + \tan^2(\chi)\right)\sigma \wedge V, \quad (3.53)$$

$$\begin{aligned}
dJ &= -\frac{1}{\cos(\chi)} \star \hat{F} - \frac{3\Lambda_2}{\cos(\chi)} e^{-A} \Psi_- + \lambda \left(\frac{4}{3} + \tan^2(\chi) \right) J \wedge \sigma \\
&\quad + \left(\Lambda_2 \tan(\chi) e^{-A} - \frac{3 + \cos^2(\chi)}{6 \cos(\chi)} Q \right) J \wedge V,
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
d\Psi_- &= -\tan(\chi) \star \hat{F} \wedge V + \left(3\Lambda_2 \tan(\chi) e^{-A} - \frac{1}{2} \cos(\chi) Q \right) V \wedge \Psi_- + \lambda \sigma \wedge \Psi_- \\
&\quad + 2\lambda \frac{\sin(\chi)}{\cos^2(\chi)} \sigma \wedge J \wedge V,
\end{aligned} \tag{3.55}$$

$$d\Psi_+ = -\tan(\chi) \hat{F} - \frac{\Lambda_2}{\cos(\chi)} e^{-A} J \wedge J + \frac{\lambda}{\cos^2(\chi)} \sigma \wedge \Psi_+ - \frac{Q}{2 \cos(\chi)} V \wedge \Psi_+. \tag{3.56}$$

The task is now to solve the supersymmetry conditions (3.40), (3.41) and (3.43)-(3.47) with internal flux (3.48) subject to the equation of motion (3.49) and Bianchi identity (2.3). We have not attempted to do this in general but we will analyze some special cases in the subsequent sections.

The above equations show that the presence of the cosmological constant Λ_2 and the phase χ introduces significant complication. We re-iterate that different normalizations of the internal spinors is necessary for having solutions with non-vanishing Λ_2 and, hence, with AdS_4 as the external space. Indeed, setting $\chi = n\pi$ for an integer n , the values corresponding to equal norms of the spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$, immediately implies from Eq. (3.52) that $\Lambda_2 = 0$. In this case, our equations reduce to the ones given in Refs. [14, 20]. We will see below that solutions with non-trivial phase χ and non-vanishing Λ_2 exist, at least locally. Also note that $\Lambda_2 = 0$ does not seem to imply, from the above equations, that the phase χ is trivial. We will later present an explicit (local) solution with vanishing Λ_2 and non-trivial χ which shows this is indeed the case.

3.4 General properties for $\Lambda_2 \neq 0$

In the case of AdS_4 as the external space we can derive a number of additional general properties by combining supersymmetry conditions with the equation of motion and Bianchi identity for \hat{F} . More specifically, consider the integrability condition for Eq. (3.47), derived by multiplying the equation with e^{2A} and taking the exterior derivative. Taking into account the Bianchi identity (3.50) as well as the relations (3.41), (3.52)-(3.56) we find

$$\Lambda_2 \left[2\hat{F} \wedge V + 2 \star \hat{F} \wedge J - 4\lambda \tan(\chi) \sigma \wedge \Psi_+ \wedge V + QV \wedge J \wedge J - \frac{2\lambda}{\cos(\chi)} \sigma \wedge J \wedge J \right] = 0. \tag{3.57}$$

Note this relation is trivially satisfied for vanishing cosmological constant, $\Lambda_2 = 0$. However, for $\Lambda_2 \neq 0$ and after inserting the explicit expression (3.48) for \hat{F} and its dual (B.64) we obtain a non-trivial constraint on the internal flux. It turns out, this constraint is satisfied if and only if

$$W = 0. \tag{3.58}$$

Note, from Eq. (3.42), this also implies that $\sigma = 0$. We have hence shown that a non-zero cosmological constant implies the absence of the vector part of the internal flux. This simplifies Eq. (3.53) to $dV = 0$. We can, therefore, introduce a coordinate system $\mathbf{y} = (z^i, y)$, where $i, j, \dots = 4, \dots, 9$ such that

$$V = dy, \tag{3.59}$$

locally. From Eqs. (3.40), (3.41) and (3.52) we then learn that all scalar quantities are independent of z^i and depend on the single coordinate y only, that is, we have

$$A = A(y), \quad \chi = \chi(y), \quad Q = Q(y), \quad c_1 = c_1(y). \tag{3.60}$$

The two internal equations (3.40) and (3.41) then read

$$c_1 = -\frac{1}{4}\sin(\chi)Q - \frac{3}{2}e^{-A}\Lambda_2, \quad (3.61)$$

$$A' = \frac{1}{6}\cos(\chi)Q, \quad (3.62)$$

where the prime denotes the derivative with respect to y . The external Eqs. (3.52)–(3.56) simplify to

$$\chi' = -\frac{1}{2}Q\sin(\chi) - 2\Lambda_2e^{-A}, \quad (3.63)$$

$$dJ = -\frac{1}{\cos(\chi)}\star\hat{F} - \frac{3\Lambda_2}{\cos(\chi)}e^{-A}\Psi_- + \left(\Lambda_2\tan(\chi)e^{-A} - \frac{3+\cos^2(\chi)}{6\cos(\chi)}Q\right)J\wedge V, \quad (3.64)$$

$$d\Psi_- = -\tan(\chi)\star\hat{F}\wedge V + \left(3\Lambda_2\tan(\chi)e^{-A} - \frac{1}{2}\cos(\chi)Q\right)V\wedge\Psi_-, \quad (3.65)$$

$$d\Psi_+ = -\tan(\chi)\hat{F} - \frac{\Lambda_2}{\cos(\chi)}e^{-A}J\wedge J - \frac{Q}{2\cos(\chi)}V\wedge\Psi_+, \quad (3.66)$$

with V given by Eq. (3.59). It is instructive to look at the components of these equations with indices in the \mathbf{z} directions only. We denote the corresponding exterior derivative in these directions by \hat{d} . Then, with the explicit expressions (3.48) and (B.64) for \hat{F} and its dual (remembering that $c_2 = 0$ in general and $W = Y = 0$ for $\Lambda_2 \neq 0$) we find

$$\hat{d}J = -\frac{1}{\cos(\chi)}[(c_1 + 3\Lambda_2e^{-A})\Psi_- + S], \quad (3.67)$$

$$\hat{d}\Psi_- = 0, \quad (3.68)$$

$$\hat{d}\Psi_+ = -\frac{1}{\cos(\chi)}\left[(\Lambda_2e^{-A} - \frac{1}{6}Q)J\wedge J + J\wedge\mathcal{A}\right]. \quad (3.69)$$

These equations can be interpreted as the differential relations for an $SU(3)$ structure, characterized by (Ψ, J) , for the six-dimensional manifolds obtained by slicing the seven-dimensional internal manifold at constant y . It follows that

$$\hat{d}\Psi_- = 0, \quad \hat{d}J\wedge J = 0, \quad (3.70)$$

the latter using Eqs. (B.41) and (B.67). These two relations are precisely the defining conditions for half-flat manifolds [33]. We, therefore, conclude that for non-zero cosmological constant, $\Lambda_2 \neq 0$, the six-dimensional slices at constant y of our internal manifold are half-flat manifolds. Note that the conclusions of this sub-section all followed from the initial result that $W = 0$. Hence, they can even be applied for vanishing cosmological constant, $\Lambda_2 = 0$, as long as one imposes the additional requirement $W = 0$.

4 Solution for $SU(3)$ singlet flux

Clearly, solving the supersymmetry conditions for general flux is quite difficult. In this section, we therefore specialize to $SU(3)$ singlet flux and consider the solutions in this case, both for vanishing and non-vanishing cosmological constant Λ_2 . To do this, we have to set the non-singlet flux components \mathcal{A} and U to zero. For $\Lambda_2 \neq 0$ we have seen previously that $W = 0$ automatically, while we impose this as an additional condition for $\Lambda_2 = 0$. In both cases the flux is given by

$$\hat{F} = -\frac{1}{6}QJ\wedge J - c_1\Psi_+\wedge V, \quad (4.71)$$

that is, it only contains the two singlet components Q and c_1 . Together with the warp factor A and the phase χ these form a set of four scalar quantities which, as we will see, are determined by a closed set of first-order differential equations. We first note, that from the external supersymmetry condition (3.40) c_1 is actually fixed in terms of the other three scalar by

$$c_1 = -\frac{1}{4}\sin(\chi)Q - \frac{3}{2}e^{-A}\Lambda_2. \quad (4.72)$$

We have mentioned earlier that the results of Section 3.4 can be applied as long as $W = 0$. In particular, we can split internal coordinates as $(y^m) = (z^i, y)$, where $i, j, \dots = 0, \dots, 9$ and write the vector V as

$$V = dy, \quad (4.73)$$

locally. The scalars Q , c_1 , A and χ are then functions of the single coordinate y only. Moreover, with Eqs. (3.62) and (3.63), we immediately have two first order differential equations for the three scalar A , Q and χ . A third equation can be obtained by inserting the flux (4.71) into the equation of motion (3.49) or the Bianchi identity (3.50) using the relation (3.52)–(3.56) to remove exterior derivatives. Both calculations actually lead to the same result, a first-order differential equation for Q . The set of first-order differential equations which determines A , Q and χ then reads

$$A' = \frac{1}{6}Q \cos(\chi), \quad (4.74)$$

$$Q' = \frac{1}{12 \cos(\chi)} [(7 + \cos^2(\chi))Q^2 - 24 \sin(\chi)\Lambda_2 e^{-A}Q - 108(\Lambda_2 e^{-A})^2], \quad (4.75)$$

$$\chi' = -\frac{1}{2}Q \sin(\chi) - 2\Lambda_2 e^{-A}. \quad (4.76)$$

With these three equations and the solutions (4.73), (4.72) for V and c_1 we have satisfied the F equation of motion, the Bianchi identity and all but three supersymmetry relations. These three remaining equations are the differential relations (3.54)–(3.56) for J and Ψ_{\pm} . They can be cast into the simple form

$$e^{-2\alpha}d(e^{2\alpha}J) = 3h\Psi_- \quad (4.77)$$

$$e^{-3\alpha}d(e^{3\alpha}\Psi_-) = 0 \quad (4.78)$$

$$e^{-3\alpha}d(e^{3\alpha}\Psi_+) = 2hJ \wedge J \quad (4.79)$$

where we have introduced the two functions

$$h = \frac{1}{3 \cos(\chi)} \left[\frac{1}{4}Q \sin(\chi) - \frac{3}{2}\Lambda_2 e^{-A} \right] \quad (4.80)$$

$$g = \frac{1}{2 \cos(\chi)} \left[\frac{1}{6}Q(1 + \cos^2(\chi)) - \Lambda_2 e^{-A} \sin(\chi) \right] \quad (4.81)$$

and α is defined by

$$\alpha' = g. \quad (4.82)$$

As before, the prime denotes the derivative with respect to y . It can be verified by straightforward computation, using the differential equations (4.74)–(4.76), that the functions h and g satisfy $h' = hg$. This implies that

$$he^{-\alpha} = c, \quad (4.83)$$

where c is an arbitrary real constant. The general solution to Eqs. (4.77)–(4.79) then takes the form

$$J = e^{-2\alpha}\omega \quad (4.84)$$

$$\Psi_{\pm} = e^{-3\alpha}\Omega_{\pm}, \quad (4.85)$$

where the forms ω and Ω_{\pm} are y -independent and satisfy the differential relations

$$\hat{d}w = 3c\Omega_{-}, \quad (4.86)$$

$$\hat{d}\Omega_{-} = 0, \quad (4.87)$$

$$\hat{d}\Omega_{+} = 2c\omega \wedge \omega. \quad (4.88)$$

We recall that \hat{d} denotes the exterior derivative with respect to the six-dimensional coordinates z^i . The forms ω and Ω_{\pm} define a six-dimensional $SU(3)$ structure with respect to a rescaled vielbein $\tilde{e}^i = \exp(\alpha)e^i$. Their above differential relations are precisely the ones for so-called nearly-Kähler manifolds [33]. The metric corresponding to these solutions therefore takes the form

$$ds^2 = e^{2A}ds_4^2 + e^{-2\alpha}ds_{(\text{nK})}^2 + dy^2, \quad (4.89)$$

where $ds_{(\text{nK})}^2$ is the metric on a nearly-Kähler manifold. The associated flux is given by

$$F = -\frac{1}{6}Qe^{-4\alpha}\omega \wedge \omega - c_1e^{-3\alpha}\Omega_{+} \wedge dy. \quad (4.90)$$

The scalar c_1 is determined by Eq. (4.72) and A , Q and χ have to be determined by solving the differential equations (4.74)–(4.76). For $c \neq 0$ the scale factor α is given by

$$e^{-\alpha} = \frac{c}{h}, \quad (4.91)$$

and is, therefore, determined once a solution for A , Q and χ has been found. Unfortunately, we did not find the general analytic solution of the system (4.74)–(4.76) and, given the complication of the equations, this may well be impossible. We have numerically integrated this system for a variety of initial conditions. All numerical solutions we have found exist on a finite interval in y only, with (curvature) singularities developing at the interval endpoints. This does not exclude the existence of special solutions, obtained from particular choices for the initial conditions in (4.74)–(4.76), which are globally defined for all y . Indeed, the case $c = 0$ which we discuss in detail in the next subsection leads to a special solution which exists on a half-infinite range in y . Unfortunately, we do not know any other special solutions to (4.74)–(4.76) which are globally defined.

4.1 A Calabi-Yau solution

The case $c = 0$ is special in that, from Eq. (4.83), it forces the function h in Eq. (4.80) to vanish identically. This leads to an additional algebraic constraint between A , Q and χ . It is easily seen that the solutions to (4.74)–(4.76) in this case are given by

$$\sin(\chi) = \frac{\Lambda_2}{k}e^{-5A} \quad (4.92)$$

$$Q = 6ke^{4A} \quad (4.93)$$

$$\alpha = A + \text{const}, \quad (4.94)$$

where k is a constant. The warp factor A is determined by the differential equation

$$A' = \sqrt{k^2 e^{8A} - (\Lambda_2)^2 e^{-2A}}. \quad (4.95)$$

The solution to this equation is only real for $A > A_{\text{crit}} = \frac{1}{5} \ln \left(\frac{\Lambda_2}{k} \right)$ and A starts at $A(y \rightarrow -\infty) = A_{\text{crit}}$. It then increases until it diverges at some finite y . One also finds from Eq. (4.92) that $\sin(\chi)$ is unity at $y \rightarrow -\infty$ and evolves to zero at the singularity. Hence, this solution exists on a half-infinite range in y , as previously claimed.

For $c = 0$ the differential relations (4.86)–(4.88) for nearly-Kähler manifolds reduce to the ones for Calabi-Yau three-folds. Hence the metric for this solution is given by

$$ds_{11}^2 = e^{2A} ds_4^2 + e^{-2A} ds_{\text{(CY)}}^2 + dy^2, \quad (4.96)$$

where $ds_{\text{(CY)}}^2$ is the Ricci-flat metric of a Calabi-Yau three-fold. The associated flux takes the form

$$\hat{F} = -k\omega \wedge \omega + 3\Lambda_2 e^{-4A} \Omega_+ \wedge dy \quad (4.97)$$

As a consistency check, we have verified that this metric and flux indeed satisfy the Einstein equations (2.1).

4.2 A special solution for vanishing cosmological constant

We have previously seen that setting the norms of the two internal spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$ equal, or, equivalently, having a trivial phase $\chi = n\pi$ for n integer, implies a vanishing cosmological constant. In this subsection we will show the opposite is not true, that is, examples with vanishing cosmological constant and non-trivial phase χ exist.

To find such an example, we consider the differential equations (4.74)–(4.76) for $\Lambda_2 = 0$. They can then be written in the simple form

$$A' = \frac{1}{6} X Q \quad (4.98)$$

$$X' = \frac{1}{2} (1 - X^2) Q \quad (4.99)$$

$$Q' = \frac{7 + X^2}{12X} Q^2 \quad (4.100)$$

where we have defined $X = \cos(\chi)$. This leads to the implicit solution

$$A = A_0 - \frac{1}{6} \ln |1 - X^2| \quad (4.101)$$

$$Q = K \left| \frac{X^7}{(1 - X^2)^4} \right|^{1/6} \quad (4.102)$$

$$e^{-2\alpha} = \frac{144c^2}{K^2} \left| \frac{1 - X^2}{X} \right|^{1/3}, \quad (4.103)$$

where A_0 and K are constants. Finally, X as a function of y can be obtained by integrating Eq. (4.99). This integral shows the solutions exist on a half-infinite range where X varies from zero at $y \rightarrow \pm\infty$ to $X = \pm 1$ at some finite value of y . At this point the solution develops a curvature singularity due to the vanishing of the scale factor (4.103).

5 Embedding into Horava-Witten theory

In this section, we would like to analyze the relation between the $SU(3)$ structure formalism and Horava-Witten theory [31, 32], that is, M-theory on the orbifold S^1/Z_2 . The action for this theory can be constructed [32] by coupling 11-dimensional supergravity to two 10-dimensional E_8 gauge multiplets which are located on the two 10-dimensional fixed planes of the orbifold. The bulk equations for this theory are identical to the ones for 11-dimensional supergravity and it is, therefore, reasonable to ask whether any of the solutions obtained so far can be embedded in Horava-Witten theory. This amounts to checking whether any of our solutions can be made to satisfy the boundary conditions of Horava-Witten theory imposed at the two 10-dimensional fixed planes. In fact, such an embedding may be quite useful to cut off curvature singularities at finite proper distance which, as we have seen, can arise in the M-theory solutions.

The most relevant boundary condition for our purposes is the one on the spinor η which parameterizes supersymmetry transformations. Let us take the coordinate in the orbifold direction to be x^\natural . In order to ensure, the various components of the gravitino remain chiral on the boundaries, η should satisfy the conditions

$$\eta = \Gamma_\natural \eta, \quad (5.104)$$

$$\eta' = -\Gamma_\natural \eta' \quad (5.105)$$

at both boundaries, where the prime denotes the derivative with respect to x^\natural . These conditions have to be analyzed for our spinor Ansatz (3.13) and we will do this subsequently for solutions with singlet flux.

5.1 Embedding of solutions with singlet flux?

We now specialize to the solutions with singlet flux found in Section 4. For these solutions one of the seven internal coordinates, previously called y , is singled out and the vector V is given by $V = dy$. It seems natural to identify the coordinate y with the orbifold direction, that is $x^\natural = y$. From Eq. (B.54) the spinor Ansatz (3.13) then takes the form

$$\eta = |\epsilon_{(1)}| \left(\theta \otimes \hat{\epsilon}_{(1)} - \lambda \gamma \theta \otimes (\gamma_\natural) \hat{\epsilon}_{(1)} \right), \quad (5.106)$$

where $\hat{\epsilon}_{(1)}$ is a normalized version of the spinor $\epsilon_{(1)}$. From Eq. (B.34) and (4.84) it follows that the two-form ω on the nearly-Kähler manifold can be written as

$$\omega_{ij} = -i \hat{\epsilon}_{(1)}^T \tilde{\gamma}_{ij} \hat{\epsilon}_{(1)}, \quad (5.107)$$

where $\tilde{\gamma}_i$ are the “curved” six-dimensional gamma matrices with respect to the nearly-Kähler metric. Given that both this metric and ω are y -independent we conclude that the same is true for the normalized spinor $\hat{\epsilon}_{(1)}$. This allows us to perform the y -derivative of η which is required for the second boundary condition (5.105). Using the spinor normalization (3.28) and (3.29) as well as various relations from Section 4 it is then straightforward to work out the boundary conditions. We find that the two conditions (5.104) and (5.105) are satisfied for singlet-flux solutions precisely if

$$\chi = (2n + 1)\pi, \quad A' = 0, \quad (5.108)$$

at both boundaries for integers n . From the differential equation (4.74) for A' the second condition is equivalent to $Q = 0$ at the boundaries. We can then numerically integrate the equations (4.74)–(4.76) starting from one of the boundaries, located at, say, $y = 0$, using the initial conditions

$\chi(0) = (2n + 1)\pi$ and $Q(0) = 0$. This guarantees the conditions (5.108) are satisfied at the first boundary. One would then hope that χ equals the same or another odd multiple of π at some value $y > 0$ which could be identified with the second boundary. The remaining free initial condition $A(0)$ may then be used to set $Q = 0$ at the second boundary. Numerical integration shows that χ either returns to the same odd multiple of π or approaches one of the neighbouring even multiples, depending on the choice of $A(0)$, but in both cases the solution terminates with a singularity. This makes it impossible to satisfy the conditions (5.108) at the second boundary. Our conclusion is, therefore, that none of the singlet flux solutions can be embedded in Horava-Witten theory. This statement can also be explicitly verified for the two special classes of solutions which we have found. Consider first the Calabi-Yau solutions of Section 4.1. The relation (4.93) implies that $A \rightarrow -\infty$ whenever $Q \rightarrow 0$ and, hence, a curvature singularity. This precludes the implementation of the boundary condition (5.108). For the solutions with vanishing Λ_2 , described in Section 4.2, we should have $X = \cos(\chi) = -1$ at the boundary. From Eq. (4.102) this means that $Q \rightarrow \infty$ which contradicts the boundary condition $Q = 0$.

5.2 Perturbative solutions

Given the complexity of the supersymmetry conditions it is clearly worthwhile asking about the existence of suitable approximation schemes and corresponding approximate solutions. An obvious such scheme is to start with a general solution for vanishing flux, that is, a direct product of four-dimensional Minkowski space and a Calabi-Yau space times S^1 as the internal space and compute first-order corrections due to a small but non-vanishing flux to this solution. In fact, a class of such approximate solutions has been found in Ref. [18, 19] in the context Horava-Witten theory. We would now like to analyze how these solutions fit into the present context of $SU(3)$ structure.

We start by reviewing the solutions in Ref. [18, 19]. The metric for these solutions has the form

$$ds_{(11)}^2 = (1 + b)dx^\mu dx^\nu \eta_{\mu\nu} + (g_{ij} + h_{ij})dz^i dz^j + (1 + \rho)dy^2. \quad (5.109)$$

Here g_{ij} , where $i, j, \dots = 4, \dots, 9$ is the Ricci-flat metric on a Calabi-Yau three-fold with Kähler form $\omega_{a\bar{b}} = -ig_{a\bar{b}}$ (a, b, \dots and \bar{a}, \bar{b}, \dots denote holomorphic and anti-holomorphic indices on the Calabi-Yau space) and $y = x^4$. Further, b , ρ and h_{ij} depend on the internal coordinates y^m and are the first-order corrections to be determined. They can be expressed in terms of a $(1, 1)$ form $B_{(2)}$ on the Calabi-Yau space satisfying the gauge condition

$$\hat{d}\hat{\star}B_{(2)} = 0. \quad (5.110)$$

The star $\hat{\star}$ denotes the six-dimensional Hodge dual with respect to the Calabi-Yau metric g_{ij} and \hat{d} is the exterior derivative with respect to the z^i coordinates. This two-form is harmonic with respect to the seven-dimensional internal space, that is, it satisfies

$$\Delta B_{(2)} + B_{(2)}'' = 0, \quad (5.111)$$

where Δ is the Laplacian on the Calabi-Yau space. In terms of this $(1, 1)$ form the metric corrections can then be written as ³

$$h_{a\bar{b}} = i \left(B_{a\bar{b}} - \frac{1}{3} \omega_{a\bar{b}} B \right), \quad (5.112)$$

³Note these relations and subsequent ones differ from the ones in Ref. [19] by a factor $\sqrt{2}$ due to a different normalization of the four-form F . For simplicity, we have also dropped various integration constants which can be absorbed into coordinate redefinitions and are irrelevant for our purposes.

$$b = \frac{1}{6}B, \quad (5.113)$$

$$\rho = -\frac{1}{3}B, \quad (5.114)$$

where

$$B = 2\omega \lrcorner B_{(2)}. \quad (5.115)$$

The associated flux can be written as

$$\star \hat{F} = dB_{(2)} = \hat{d}B_{(2)} + B'_{(2)} \wedge dy, \quad (5.116)$$

where \star refers to the seven-dimensional Hodge dual and the prime denotes the derivative with respect to y . The corresponding spinor in the supersymmetry transformation reads

$$\zeta_+ = e^{-\phi} \zeta \quad (5.117)$$

where ζ is the covariantly constant complex spinor on the Calabi-Yau space satisfying

$$\gamma_{\bar{a}} \zeta = 0, \quad \gamma_{\mathfrak{z}} \zeta = \zeta, \quad \zeta^\dagger \zeta = 1. \quad (5.118)$$

The scalar ϕ encodes the first-order correction to the spinor and is explicitly given by

$$\phi = -\frac{1}{24}B. \quad (5.119)$$

Our notation suggests that the spinor in Eq. (5.117) should be identified with the complex spinor ζ_+ which appeared in the general formalism. That this is indeed sensible can be seen from the general projection conditions (B.21) which imply the relations (5.118). This completes the review of the solutions.

We would now like to check that these solutions indeed satisfy the general supersymmetry conditions summarized in Section 3.3. This provides a useful consistency check for the general results in this paper as well as explicit (approximate) expressions for the forms V , J and Ψ defining the $SU(3)$ structure. We first remark that the flux (5.116) clearly satisfies the F equations of motion (3.49) and Bianchi identity (3.50) to leading order as a result of the properties (5.110) and (5.111).

Next we should identify the quantities which enter the general formalism in terms of the approximate solutions. Note first that we are working with four-dimensional Minkowski space and, hence,

$$\Lambda_2 = 0. \quad (5.120)$$

Further, from Eqs. (5.117) and (5.118) we have $0 = \zeta_+^T \gamma_{ab} \zeta_+ = 2g_{ab} \zeta_+^T \zeta_+$. Using Eq. (B.16), we conclude that $\tilde{\Xi} = 0$ which implies, by virtue of Eq. (3.29), that

$$\sin(\chi) = 0. \quad (5.121)$$

We are, therefore, dealing with solutions where the two norms of the Majorana spinors are identical. By comparing the metric Ansätze (5.109) and (3.6) we identify the warp factor to leading order as

$$A = \frac{b}{2} = \frac{B}{12}. \quad (5.122)$$

For the singlet and vector components of the flux one finds

$$Q = -\frac{1}{2}B', \quad (5.123)$$

$$c_1 = 0, \quad (5.124)$$

$$\sigma = \frac{1}{4}\hat{d}B. \quad (5.125)$$

Finally, one has to compute the approximate SU(3) structure (V, J, Ψ) which can be done using the bi-spinor expressions in Appendix B and inserting the spinor (5.117) and the internal metric from Eqs. (5.109) and (5.112). The result to first order in the flux is

$$V = \left(1 - \frac{B}{6}\right) dy, \quad (5.126)$$

$$J = \omega + B_{(2)} - \frac{1}{3}B\omega, \quad (5.127)$$

$$\Psi = \Omega - \frac{1}{4}B\Omega, \quad (5.128)$$

where Ω is the $(3, 0)$ form on the Calabi-Yau space. With these identifications it is straightforward to verify that the supersymmetry conditions (3.40), (3.41) and (3.43)–(3.47) are indeed satisfied to first order in the flux, provided we choose the phase χ so that $\cos(\chi) = -1$.

Are these the only perturbative supersymmetric solutions obtained from zeroth order solutions which are direct products of four-dimensional Minkowski space, a circle and a Calabi-Yau three-fold? The answer is yes as long as one requires that the cosmological constant remains zero under the perturbations⁴. To verify this statement, first note that from Eq. (3.40) we have $c_1 = 0$ since $\Lambda_2 = 0$ and $\sin(\chi) = 0$ to zeroth order. This means that (5.116) represents the most general flux where $B_{(2)}$ is a $(1, 1)$ form. One can insert this flux into Eqs. (3.41), (3.44)–(3.47) and integrate these equations to obtain precisely the scale factor (5.122) and the expressions (5.126)–(5.128) for the SU(3) structure. The warped Calabi-Yau solutions of Ref. [18, 19] are therefore the only perturbative solutions of Horava-Witten theory starting from the direct product of Calabi-Yau space, circle and Minkowski space at lowest order. It is not clear to us whether or not there exist other solutions of Horava-Witten, corresponding to $N = 1$ four-dimensional supersymmetry, which cannot be obtained in this perturbative manner. We have shown for the singlet-flux solutions of Section 4 that an embedding into Horava-Witten theory is not possible. However, there may still be other solutions with a more complicated structure of the flux which do allow for such an embedding.

6 Conclusion

In this paper, we have analyzed supersymmetric solutions of M-theory based on a seven-dimensional internal space with SU(3) structure and a four-dimensional maximally symmetric space, either four-dimensional Minkowski space or AdS₄. We have worked with what we believe is the most general Ansatz for the fields and particularly the supersymmetry spinor and have derived the complete set of supersymmetry conditions in this case. These conditions are summarized in Section 3.3. They are more general and more complicated than related equations previously derived in the literature since we have allowed the norm of the two spinors which define the SU(3) structure to be different.

⁴This means that the topology of the four-dimensional space should not change when including the first-order corrections, an assumption which seems reasonable in a perturbative calculation.

This introduces a new degree of freedom into the problem which can be conveniently parameterized by a phase χ and which has been ignored in earlier work, leading to incorrect conclusions. When χ is set to one of its trivial values $\chi = n\pi$, corresponding to equal norms of the two spinors, the equations of Section 3.3 reduce to the ones previously derived [14, 20].

We have obtained a number of general results which must hold for all supersymmetric M-theory solutions of the type considered in this paper. Specifically, we have found that the external flux in the direction of the four-dimensional maximally symmetric space and c_2 , one of the three singlets contained in the internal flux, always vanish. Further, we have seen that from the two types of Killing spinors on AdS_4 (see Eq. 3.14) only one type can be compatible with supersymmetry and that the associated cosmological constant Λ_2 must be independent of the internal coordinates. It also turned out that having different norms of the two spinors, or equivalently, having a nontrivial phase $\chi \neq n\pi$, is a necessary condition for the existence of solutions with non-zero cosmological constant and, hence, AdS_4 as the external space.

For non-vanishing cosmological constant, $\Lambda_2 \neq 0$, we have arrived at a number of additional general conclusions. We found that in this case the vector part W of the internal flux always vanishes. This implied the vector V , which together with the two-form J and the complex three-form Ψ defines the seven-dimensional $\text{SU}(3)$ structure, could be written as $V = dy$ for some internal coordinate y . Moreover, all singlet quantities, namely the two singlet flux components Q and c_1 as well as the warp factor A and the phase χ , turned out to be functions of y only. We also showed that the six-dimensional spaces, obtained by slicing the internal space at constant y , are always half-flat manifolds.

As an application of our results, we have considered the case of singlet flux where only the two $\text{SU}(3)$ singlets Q and c_1 of the internal flux are non-vanishing. We showed the internal seven-dimensional space is a warped product of a circle with coordinate y and a six-dimensional nearly-Kähler manifold, in this case. The problem of finding explicit solutions has been reduced to the one of finding solutions to a complicated system of three first-order differential equations for the warp factor A , the singlet flux Q and the phase χ . Special solutions include a case where the nearly-Kähler manifold degenerates to a Calabi-Yau three-fold but $\Lambda_2 \neq 0$ and another case with $\Lambda_2 = 0$, a nearly-Kähler manifold and a non-trivial phase χ . The latter example shows that a vanishing cosmological constant does not imply a trivial phase χ . Hence, even for Minkowski space as the external space our equations are more general than the ones obtained for equal spinor norms. We did not find the general analytic solution for the first-order differential equations which determine A , Q and χ . Numerically solutions exist on a finite interval in y with the interval endpoints corresponding to curvature singularities. This, however, does not exclude the existence of globally defined solutions which may arise from special initial values. Indeed, the two special solutions mentioned above exist on a half-infinite range in y rather than an interval. It would be interesting to study the global properties of the differential equations (4.74)–(4.76) to see whether solutions defined for all y do or do not exist.

Finally, we have discussed the embedding of solutions into Horava-Witten theory. This problem is of some phenomenological relevance since such solutions would lead to heterotic vacua with $N = 1$ supersymmetry in four dimensions. Unfortunately, we found none of the singlet-flux solutions could be made to satisfy the appropriate heterotic boundary conditions. As a consistency check we have also verified that the approximate warped Calabi-Yau solutions of Horava-Witten theory described in Ref. [18, 19] satisfy our general equations. We have also shown they are the only perturbative solutions of this theory based on a lowest-order space which is a direct product of four-dimensional

Minkowski space, a Calabi-Yau three-fold and a circle. It would be interesting to see whether there exist any exact solutions for a more complicated structure of the flux which satisfy heterotic boundary conditions.

Acknowledgments We would like to thank James Sparks and Daniel Waldram for discussions. Both authors are supported by a PPARC Advanced Fellowship.

Appendix

A Gamma matrices and spinors in $d = 4, 7, 11$

In this section, we summarize our conventions for gamma matrices and spinors in dimensions four, seven and eleven. We begin with four dimensions.

Four-dimensional curved indices are denoted by $\mu, \nu, \dots = 0, 1, 2, 3$ and their flat counterparts by $\underline{\mu}, \underline{\nu}, \dots = 0, 1, 2, 3$. We adopt the convention that explicit indices refer to tangent space indices. The gamma matrices $\gamma_{\underline{\mu}}$ are 4×4 matrices satisfying the standard anti-commutation relations

$$\{\gamma_{\underline{\mu}}, \gamma_{\underline{\nu}}\} = 2\eta_{\underline{\mu}\underline{\nu}} \quad (\text{A.1})$$

where $(\eta_{\underline{\mu}, \underline{\nu}}) = \text{diag}(-1, +1, +1, +1)$ is the “mostly plus” Minkowski metric. Their hermitian conjugate $(\gamma_{\underline{\mu}})^\dagger$ is given by

$$(\gamma_{\underline{\mu}})^\dagger = \gamma_0 \gamma_{\underline{\mu}} \gamma_0 . \quad (\text{A.2})$$

We normalize the Levi-Civita tensor $\epsilon_{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}}$ by $\epsilon_{0123} = 1$ and define

$$\gamma \equiv \frac{i}{4!} \epsilon_{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}} \gamma^{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}} = -i\gamma_0\gamma_1\gamma_2\gamma_3 , \quad (\text{A.3})$$

where a γ with multiple indices denotes the anti-symmetrized product of gamma matrices, as usual. Clearly, we have $\gamma^2 = \mathbf{1}_4$. The generators of the four-dimensional Lorentz group are given by

$$\sigma_{\underline{\mu}\underline{\nu}} = \frac{1}{4} [\gamma_{\underline{\mu}}, \gamma_{\underline{\nu}}] = \frac{1}{2} \gamma_{\underline{\mu}\underline{\nu}} \quad (\text{A.4})$$

For convenience, we will use a real representation of the gamma matrices $\gamma_{\underline{\mu}}$ so that γ in Eq. (A.3) is purely imaginary. Then, spinors $\theta \in \mathbb{C}^4$ are Majorana precisely when they are real. Weyl-spinors are characterized by $\gamma\theta = \pm\theta$, as usual.

Let us now turn to seven dimensions. As curved indices we take $m, n, \dots = 4, \dots, 9, \natural$ with flat counterparts $\underline{m}, \underline{n}, \dots = 4, \dots, 9, \natural$. As before, explicit indices refer to tangent space indices. The gamma matrices $\gamma_{\underline{m}}$ are hermitian 8×8 matrices satisfying

$$\{\gamma_{\underline{m}}, \gamma_{\underline{n}}\} = 2\delta_{\underline{m}\underline{n}} . \quad (\text{A.5})$$

The normalization is fixed by the relation

$$\gamma_4 \dots \gamma_9 \gamma_{\natural} = -i\mathbf{1}_8 . \quad (\text{A.6})$$

The generators of $\text{SO}(7)$ are given by

$$\sigma_{\underline{mn}} = \frac{1}{4}[\gamma_{\underline{m}}, \gamma_{\underline{n}}] = \frac{1}{2}\gamma_{\underline{mn}}. \quad (\text{A.7})$$

We will use a purely imaginary representation of the gamma matrices, $\gamma_{\underline{m}}^* = -\gamma_{\underline{m}}$, so that

$$(\gamma_{\underline{m}})^T = -\gamma_{\underline{m}}, \quad (\gamma_{\underline{mn}})^T = -\gamma_{\underline{mn}}, \quad (\gamma_{\underline{mnp}})^T = \gamma_{\underline{mnp}}. \quad (\text{A.8})$$

In this representation, spinors $\zeta \in \mathbb{C}^8$ are Majorana precisely when they are real.

Finally, in eleven dimensions curved indices are denoted by $M, N, \dots = 0, \dots, 9, \natural$ with flat counterparts $\underline{M}, \underline{N}, \dots = 0, \dots, 9, \natural$. Explicit indices refer to tangent space indices. The gamma matrices are 32×32 matrices $\Gamma_{\underline{M}}$ satisfying

$$\{\Gamma_{\underline{M}}, \Gamma_{\underline{N}}\} = 2\eta_{\underline{MN}}, \quad (\text{A.9})$$

where $(\eta_{\underline{MN}}) = (-1, +1, \dots, +1)$ is the “mostly-plus” Minkowski metric. For the hermitian conjugate we have

$$(\Gamma_{\underline{M}})^\dagger = \Gamma_0 \Gamma_{\underline{M}} \Gamma_0, \quad (\text{A.10})$$

and the normalization is such that

$$\Gamma_0 \dots \Gamma_9 \Gamma_{\natural} = \mathbf{1}_{32}. \quad (\text{A.11})$$

The generators of the 11-dimensional Lorentz group read

$$\Sigma_{\underline{MN}} = \frac{1}{4}[\Gamma_{\underline{M}}, \Gamma_{\underline{N}}] = \frac{1}{2}\Gamma_{\underline{MN}}. \quad (\text{A.12})$$

For convenience we use a real representation of the gamma matrices $\Gamma_{\underline{M}}$. Then, spinors $\eta \in \mathbb{C}^{32}$ are Majorana if they are real. Such a real representation can be obtained in terms for the four-dimensional and seven-dimensional gamma matrices introduced above by setting

$$\Gamma_{\underline{\mu}} = \gamma_{\underline{\mu}} \otimes \mathbf{1}_8, \quad \Gamma_{\underline{m}} = \gamma \otimes \gamma_{\underline{m}}. \quad (\text{A.13})$$

Note these matrices are indeed real given that $\gamma_{\underline{\mu}}$ are real and $\gamma_{\underline{m}}$ and γ are purely imaginary.

B $\text{SU}(3)$ structures in seven dimensions

In this Appendix we discuss the general formalism dealing with seven-dimensional $\text{SU}(3)$ structure manifolds [14, 20]. These manifolds form the “internal” part of the supersymmetric solutions of M-theory considered in this paper.

Consider a seven-dimensional spin manifold X with two Majorana spinors $\epsilon_{(i)}$, where $i = 1, 2$, linear independent at each point. The existence of these two spinors implies the reduction of the generic structure group $\text{Spin}(7)$ of the spin bundle to $\text{SU}(3)$. The spinors then correspond to the two singlets in the decomposition of the **8** representation of $\text{Spin}(7)$ under $\text{SU}(3)$. It is such seven-dimensional manifolds with two spinors $\epsilon_{(i)}$ and structure group $\text{SU}(3)$ which we are interested in.

Alternatively, one can discuss the reduction of the structure group in two steps. A single Majorana spinor $\epsilon_{(1)}$ reduces $\text{Spin}(7)$ to a G_2 subgroup, implying the existence of a G_2 structure. This spinor corresponds to the singlet in the decomposition of the **8** representation of $\text{Spin}(7)$ under

G_2 . A second, independent spinor $\epsilon_{(2)}$ leads to another G_2 subgroup of $\text{Spin}(7)$ in the same way. These two G_2 subgroups intersect in $\text{SU}(3)$, which is the relevant structure group in the presence of two independent spinors.

Manifolds with $\text{SU}(3)$ structure can also be characterized by a collection of $\text{SU}(3)$ -invariant forms which can be obtained as bi-linears of the spinors $\epsilon_{(i)}$. To this end, it is useful to define the complex spinors

$$\zeta_{\pm} = \epsilon_{(1)} \pm i\epsilon_{(2)}. \quad (\text{B.14})$$

We can now introduce the following forms

$$\Xi = \zeta_+^\dagger \zeta_+, \quad (\text{B.15})$$

$$\tilde{\Xi} = \zeta_+^\dagger \zeta_-, \quad (\text{B.16})$$

$$\Xi_m = -\zeta_+^\dagger \gamma_m \zeta_+ = \zeta_-^\dagger \gamma_m \zeta_-, \quad (\text{B.17})$$

$$\Xi_{mn} = i\zeta_+^\dagger \gamma_{mn} \zeta_+ = -i\zeta_-^\dagger \gamma_{mn} \zeta_-, \quad (\text{B.18})$$

$$\Xi_{mnp} = i\zeta_+^\dagger \gamma_{mnp} \zeta_+, \quad (\text{B.19})$$

$$\tilde{\Xi}_{mnp} = -\zeta_+^\dagger \gamma_{mnp} \zeta_-, \quad (\text{B.20})$$

where we have used seven-dimensional curved indices ⁵ $m, m, \dots = 1, \dots, 7$. Their flat counterparts will be denoted by $\underline{m}, \underline{n}, \dots = 1, \dots, 7$ and explicit indices always refer to tangent space indices. Although it is possible to work with these spinor bi-linears while leaving $\epsilon_{(i)}$ general, it is much more useful, following [28, 24, 15], to pick a particular basis where the spinors satisfy a series of projection conditions. A convenient choice for these conditions is provided by

$$\gamma_{1357}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma_{1357}\epsilon_{(2)} = \epsilon_{(2)}, \quad (\text{B.21})$$

$$\gamma_{1256}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma_{1256}\epsilon_{(2)} = -\epsilon_{(2)}, \quad (\text{B.22})$$

$$\gamma_{1234}\epsilon_{(1)} = -\epsilon_{(1)}, \quad \gamma_{1234}\epsilon_{(2)} = -\epsilon_{(2)}. \quad (\text{B.23})$$

Note each of the three conditions for a given spinor projects onto a four-dimensional subspace and, together, they restrict the spinor to a one-dimensional subspace. Further, as we will see, the conditions are chosen so that the two one-dimensional subspaces for $\epsilon_{(1)}$ and $\epsilon_{(2)}$ are orthogonal. We specify the frame in which the above conditions hold by vielbein one-forms $e^{\underline{m}}$ and also introduce three complex one forms ξ^a , where $a, b, \dots = 1, 2, 3$, and their complex conjugates $\xi^{\bar{a}}$, where $\bar{a}, \bar{b}, \dots = \bar{1}, \bar{2}, \bar{3}$ by

$$\xi^1 = e^1 + ie^2, \quad (\text{B.24})$$

$$\xi^2 = e^3 + ie^4, \quad (\text{B.25})$$

$$\xi^3 = e^5 + ie^6. \quad (\text{B.26})$$

It is easy to see that the spinor $\gamma_{12}\epsilon_{(1)}$ satisfies the same projection conditions as $\epsilon_{(2)}$. Hence, we must have

$$\epsilon^2 = \lambda \gamma_{12}\epsilon^1, \quad (\text{B.27})$$

for a real function λ . It is clear that λ determines the relative normalization of the two spinors, that is,

$$|\epsilon_{(2)}|^2 = \lambda^2 |\epsilon_{(1)}|^2, \quad (\text{B.28})$$

⁵In this Appendix we only deal with seven-dimensional objects and we will, unlike in the rest of the paper, use the index range $1, \dots, 7$, for simplicity. If results in this Appendix are used in the general part of the paper, the indices should be mapped according to $(1, \dots, 7) \rightarrow (4, \dots, 9, \mathfrak{q})$.

with the norm $|\cdot|$ defined by

$$|\epsilon_{(i)}|^2 = \epsilon_{(i)}^T \epsilon_{(i)} . \quad (\text{B.29})$$

Another important conclusion from Eq. (B.27) is that

$$\epsilon_{(1)}^T \epsilon_{(2)} = 0 , \quad (\text{B.30})$$

since, from Eq. (A.8), γ_{12} is an anti-symmetric matrix. This confirms that the two spinors are indeed orthogonal as stated above.

The above projection conditions can be used to systematically express the forms (B.15)–(B.20) in terms of the vielbein one-forms $e^{\underline{m}}$. One finds

$$\Xi = (1 + \lambda^2) |\epsilon_{(1)}|^2 , \quad (\text{B.31})$$

$$\tilde{\Xi} = (1 - \lambda^2) |\epsilon_{(1)}|^2 , \quad (\text{B.32})$$

$$\Xi_{(1)} = 2\lambda |\epsilon_{(1)}|^2 V , \quad (\text{B.33})$$

$$\Xi_{(2)} = 2\lambda |\epsilon_{(1)}|^2 J , \quad (\text{B.34})$$

$$\Xi_{(3)} = |\epsilon_{(1)}|^2 [(1 - \lambda^2) \Psi_- - (1 + \lambda^2) J \wedge V] , \quad (\text{B.35})$$

$$\tilde{\Xi}_{(3)} = 2\lambda |\epsilon_{(1)}|^2 \Psi_+ + i |\epsilon_{(1)}|^2 [(1 + \lambda^2) \Psi_- - (1 - \lambda^2) J \wedge V] , \quad (\text{B.36})$$

where the forms V , J and Ψ are defined as

$$V = e^7 , \quad (\text{B.37})$$

$$J = \frac{i}{2} (\xi^1 \wedge \xi^{\bar{1}} + \xi^2 \wedge \xi^{\bar{2}} + \xi^3 \wedge \xi^{\bar{3}}) , \quad (\text{B.38})$$

$$\Psi = \Psi_+ + i\Psi_- = \xi^1 \wedge \xi^2 \wedge \xi^3 . \quad (\text{B.39})$$

We have initially set up the seven-dimensional $\text{SU}(3)$ structure by postulating the existence of two Majorana spinors $\epsilon_{(1)}$ and $\epsilon_{(2)}$. Alternatively, the $\text{SU}(3)$ structure can now be characterized by a one-form V , a two-form J and a complex three-form Ψ which, in terms of the vielbein $e^{\underline{m}}$ take the form (B.37)–(B.39). The relation between those two approaches of characterizing an $\text{SU}(3)$ structure is, of course, provided by the above Eqs. (B.15)–(B.20) and (B.31)–(B.36). It is easy to see from (B.37)–(B.39) that V , J and Ψ satisfy

$$\Psi \wedge \bar{\Psi} = -\frac{4i}{3} J \wedge J \wedge J , \quad (\text{B.40})$$

$$\Psi \wedge J = 0 , \quad (\text{B.41})$$

$$V \lrcorner J = 0 , \quad (\text{B.42})$$

$$V \lrcorner \Psi = 0 , \quad (\text{B.43})$$

$$V \lrcorner V = 1 , \quad (\text{B.44})$$

$$J^m{}_n J^n{}_p = -\delta_p^m + V^m V_p , \quad (\text{B.45})$$

$$J_m{}^n \Psi_{\pm npq} = \mp \Psi_{\mp mpq} , \quad (\text{B.46})$$

$$\star \Psi_{\pm} = \pm \Psi_{\mp} \wedge V , \quad (\text{B.47})$$

$$\star(J \wedge V) = \frac{1}{2} J \wedge J . \quad (\text{B.48})$$

Here, \lrcorner denotes the contraction of two forms or, more precisely,

$$(\alpha \lrcorner \beta)_{i_1 \dots i_n} = \frac{1}{p!} \alpha^{j_1 \dots j_p} \beta_{j_1 \dots j_p i_1 \dots i_n} , \quad (\text{B.49})$$

for a p -form α and a $(p+n)$ -form β . Note that from Eq. (B.45), the tensor J^m_n can be viewed as an almost complex structure (suitably generalized to seven dimensions by including the vector V). It can be used to introduce holomorphic and anti-holomorphic indices in the usual way. Let us refer to a form with q holomorphic, r anti-holomorphic and $s \in \{0, 1\}$ singlet indices (in the direction of V) as a (q, r, s) form. In this terminology, V is a $(0, 0, 1)$ form, J is $(1, 1, 0)$ form and Ψ is a $(3, 0, 0)$ form, the latter as a result of Eq. (B.46).

Let us briefly digress to explain the precise relation between the above $SU(3)$ structure and the two G_2 structures associated with $\epsilon_{(1)}$ and $\epsilon_{(2)}$. To this end we introduce the two three-forms

$$\Phi_{mnp}^{(i)} = i\epsilon_{(i)}^T \gamma_{mnp} \epsilon_{(i)} , \quad (B.50)$$

where $i = 1, 2$. Using the projection conditions (B.21) it is straightforward to show that

$$\Phi^{(i)} = |\epsilon_{(i)}|^2 \phi^{(i)} , \quad (B.51)$$

where

$$\phi^{(1)} = \Psi_- - J \wedge V = \text{Im}(\xi^1 \wedge \xi^2 \wedge \xi^3) - \frac{i}{2}(\xi^1 \wedge \xi^{\bar{1}} + \xi^2 \wedge \xi^{\bar{2}} + \xi^3 \wedge \xi^{\bar{3}}) \wedge e^7, \quad (B.52)$$

$$\phi^{(2)} = -\Psi_- - J \wedge V = -\text{Im}(\xi^1 \wedge \xi^2 \wedge \xi^3) - \frac{i}{2}(\xi^1 \wedge \xi^{\bar{1}} + \xi^2 \wedge \xi^{\bar{2}} + \xi^3 \wedge \xi^{\bar{3}}) \wedge e^7. \quad (B.53)$$

The two G_2 structures can either be characterized by the two spinors $\epsilon_{(i)}$ or the two three-forms $\phi^{(i)}$ in Eqs. (B.52) and (B.53) and the relation between these two viewpoints is provided by Eqs. (B.50) and (B.51).

When we come to study the conditions that supersymmetry places on the bi-linears it will be useful to have some of the above equations for the spinors $\epsilon_{(i)}$ available in a covariant form. For example, with the help of the projections (B.21) it is easy to see that the relation (B.27) between $\epsilon_{(1)}$ and $\epsilon_{(2)}$ can be covariantly written as

$$\epsilon^2 = i\lambda V^m \gamma_m \epsilon^1. \quad (B.54)$$

For the action of gamma matrices on the spinors we find the covariant results

$$\gamma^{mn} \epsilon_{(1)} = i(J \wedge V)^{mnp} \gamma_p \epsilon_{(1)} - i\Psi_-^{mnp} \gamma_p \epsilon_{(1)}, \quad (B.55)$$

$$\begin{aligned} \gamma^{mnp} \epsilon_{(1)} &= i(J \wedge V)^{mnp} \epsilon_{(1)} - i\Psi_-^{mnp} \epsilon_{(1)} + (\Psi_+ \wedge V)^{mnpq} \gamma_q \epsilon_{(1)} \\ &\quad + \frac{1}{2}(J \wedge J)^{mnpq} \gamma_q \epsilon_{(1)}, \end{aligned} \quad (B.56)$$

$$\begin{aligned} \gamma^{mnpq} \epsilon_{(1)} &= -\frac{1}{2}(J \wedge J)^{mnpq} \epsilon_{(1)} - (\Psi_+ \wedge V)^{mnpq} \epsilon_{(1)} - 4i(J \wedge V)^{[mnp} \gamma^{q]} \epsilon_{(1)} \\ &\quad + 4i\Psi_-^{[mnp} \gamma^{q]} \epsilon_{(1)}, \end{aligned} \quad (B.57)$$

$$\gamma^{mnpqr} \epsilon_{(1)} = -5(\Psi_+ \wedge V)^{[mnpq} \gamma^{r]} \epsilon_{(1)} - \frac{5}{2}(J \wedge J)^{[mnpq} \gamma^{r]} \epsilon_{(1)}. \quad (B.58)$$

The $SU(3)$ structure defined by (Ψ, J, V) can be used to decompose an anti-symmetric tensor field on the seven-dimensional manifold into its various $SU(3)$ parts. For us, the relevant case is that of a four-form and we will briefly explain the relevant details [20] in this case. A four-form

Name	SU(3) representation	index type	constraints
Q, c_1, c_2	1	(0, 0, 0)	none
Y	3 + 3	(1, 0, 0), (0, 1, 0)	$V \lrcorner Y = 0$
W	3 + $\bar{\mathbf{3}}$	(1, 0, 0), (0, 1, 0)	$V \lrcorner W = 0$
\mathcal{A}	8	(1, 1, 0)	$\mathcal{A} \lrcorner \Psi = 0$ $J \lrcorner \mathcal{A} = 0$ $V \lrcorner \mathcal{A} = 0$
U	6 + $\bar{\mathbf{6}}$	(2, 1, 0), (1, 2, 0)	$\Psi \lrcorner U = 0$ $J \lrcorner U = 0$ $V \lrcorner U = 0$

Table 1: Various SU(3) components of a four-form field on a seven-dimensional manifold.

in seven dimensions transforms in the $\mathbf{35}_{\text{SO}(7)}$ representation of the generic structure group SO(7). Under SU(3) this representation decomposes as

$$\mathbf{35}_{\text{SO}(7)} \rightarrow [3 \mathbf{1} + 2 (\mathbf{3} + \bar{\mathbf{3}}) + (\mathbf{6} + \bar{\mathbf{6}}) + \mathbf{8}]_{\text{SU}(3)} . \quad (\text{B.59})$$

We denote the three singlets by Q , c_1 and c_2 and the two vectors in $\mathbf{3} + \bar{\mathbf{3}}$ as Y and W . They are characterized by the constraints

$$V \lrcorner Y = V \lrcorner W = 0 . \quad (\text{B.60})$$

Both Y and W consist of a (1, 0, 0) and a (0, 1, 0) part. Further, the octet in the above decomposition can then be represented by a traceless (1, 1, 0) form which we denote by \mathcal{A} . This two-form \mathcal{A} can alternatively be characterized by the coordinate-independent constraints

$$J \lrcorner \mathcal{A} = 0 , \quad \mathcal{A} \lrcorner \Psi = 0 , \quad V \lrcorner \mathcal{A} = 0 . \quad (\text{B.61})$$

Likewise, the $\mathbf{6} + \bar{\mathbf{6}}$ part can be represented by a three-form U which contains a (2, 1, 0) and a (1, 2, 0) part and is characterized by

$$\Psi \lrcorner U = 0 , \quad J \lrcorner U = 0 , \quad V \lrcorner U = 0 . \quad (\text{B.62})$$

For convenience, we have summarized these components and their properties in Table 1. A general four-form \hat{F} can now be expressed as

$$\hat{F} = -\frac{1}{6} Q J \wedge J + J \wedge \mathcal{A} + \Psi_- \wedge Y - c_1 \Psi_+ \wedge V - c_2 \Psi_- \wedge V + V \wedge J \wedge W + V \wedge U . \quad (\text{B.63})$$

Using the explicit expressions (B.37)–(B.39) for V , J and Ψ along with the constraints listed in Table 1 one obtains the Hodge-dual

$$\star \hat{F} = -\frac{1}{3} Q J \wedge V - \mathcal{A} \wedge V - V \wedge (Y \lrcorner \Psi_+) + c_1 \Psi_- - c_2 \Psi_+ + J \wedge (W \lrcorner J) + S , \quad (\text{B.64})$$

where S is a three-form defined by

$$S = \star(V \wedge U) . \quad (\text{B.65})$$

It contains a (2, 1, 0) and a (1, 2, 0) part and can be characterized as a three-form satisfying the constraints

$$\Psi \lrcorner S = 0 , \quad J \lrcorner S = 0 , \quad V \lrcorner S = 0 . \quad (\text{B.66})$$

The second of these relations implies that $S \lrcorner (J \wedge J) = 0$ and using Eq. (B.48) we conclude that

$$S \wedge J = 0 . \quad (\text{B.67})$$

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